AN AF ALGEBRA ASSOCIATED WITH THE FAREY TESSELLATION

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Abstract. To the Farey tessellation of the upper half-plane we associate an AF algebra $\mathfrak{A}$ encoding the cutting sequences that define vertical geodesics. The Effros-Shen AF algebras arise as quotients of $\mathfrak{A}$. Using the path algebra model for AF algebras we construct, for each $\tau \in (0, \frac{1}{4}]$, projections (E_n) in $\mathfrak{A}$ such that $E_n E_{n\pm 1} E_n \le \tau E_n$.

Contents

Introduction

The semigroup $\mathfrak{S}$ generated by the matrices $A = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ is isomorphic to $\mathbb{F}_2^+$, the free semigroup on two generators. This fact, intimately connected to the continued fraction algorithm, can be visualized by means of the *Farey tessellation* $\{g\mathbb{G} : g \in \mathfrak{S}\}$ of $\mathbb{H}$ depicted in Figure 1, where $\mathbb{G} = \left\{0 \le \Re z \le 1 : \left|z - \frac{1}{2}\right| \ge \frac{1}{2}\right\}$ (cf., e.g., [25]).

The half-strip $0 \le \Re z \le 1$, $\Im z > 0$, is tessellated precisely by the images of $\mathbb{G}$ under matrices from the set

$$\mathfrak{S}_* = \{I\} \cup \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2(\mathbb{Z}) : 0 \le a \le c,\ 0 \le b \le d \right\}.$$

By suspending the cusps in this tessellation (which correspond to rational numbers in $[0, 1]$) with appropriate (infinite) multiplicities, one gets the diagram $\mathcal{G}$ from Figure 2 (cf. [19]). This diagram reflects both the elementary mediant construction, that produces from a pair $(\frac{p}{q}, \frac{p'}{q'})$ of rational numbers with $p'q - pq' = 1$ the new pairs $(\frac{p}{q}, \frac{p+p'}{q+q'})$ and $(\frac{p+p'}{q+q'}, \frac{p'}{q'})$ with the same property, and the geometry of the continued fraction algorithm. As in the case of the Pascal triangle, in $\mathcal{G}$ one writes the sum of the

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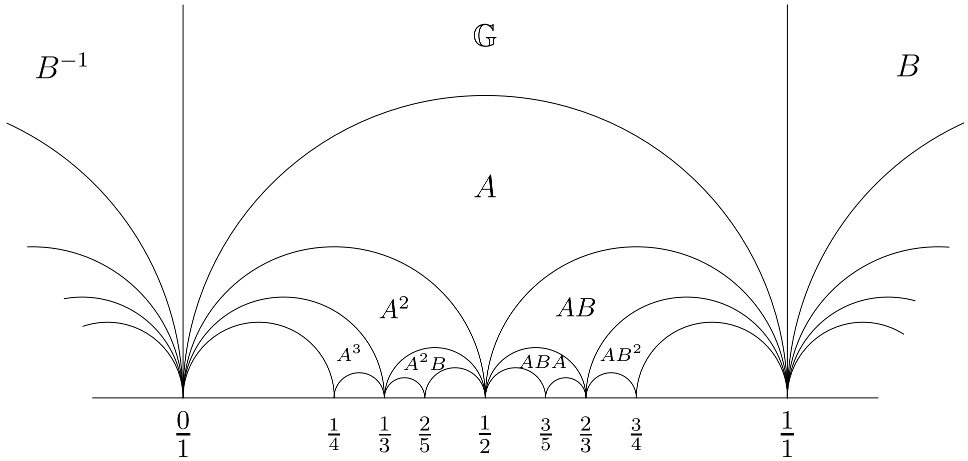


FIGURE 1. The Farey tessellation

denominators of two neighbors from the same floor into the next floor of the diagram. One keeps, however, a copy of each denominator at the next floor. For this reason, such a diagram was called the *Pascal triangle with memory* [18]. There is a remarkable one-to-one correspondence between the integer solutions of the equation $ad - bc = 1$ with $0 \leq a \leq c$, $0 \leq b \leq d$, and the rational labels of two neighbors at the same floor in $\mathcal{G}$, acquired by the mediant construction and by keeping each label at the next floor in the diagram.

The thrust of this paper is the remark that, by regarding $\mathcal{G}$ as a Bratteli diagram, one gets an AF algebra $\mathfrak{A} = \varinjlim \mathfrak{A}_n$ with interesting properties. This algebra is closely related with the *Effros–Shen AF algebras* [10, 21] which we show to arise as primitive quotients of $\mathfrak{A}$. The primitive ideal space $\operatorname{Prim} \mathfrak{A}$ is identified with the disjoint union of the irrational numbers in $[0, 1]$ and three copies of the rational ones, except for the endpoints 0 and 1 which are represented by only two copies.

In [3] it was shown that any separable abelian C^*-algebra $\mathfrak{Z}$ is the center $Z(\mathcal{A})$ of an AF algebra $\mathcal{A}$. The AF algebra $\mathfrak{A}$ can actually be retrieved from that abstract construction by embedding $\mathfrak{Z} = C[0, 1]$ into the norm closure in $L^\infty[0, 1]$ of the linear space of the characteristic functions of open sets $(\frac{k}{2^n}, \frac{k+1}{2^n})$ and of singleton sets $\{\frac{\ell}{2^n}\}$, $n \geq 0$, $0 \leq k < 2^n$, $0 \leq \ell \leq 2^n$. In particular this shows that $Z(\mathfrak{A}) = C[0, 1]$.

The connecting maps $K_0(\mathfrak{A}_n) \hookrightarrow K_0(\mathfrak{A}_{n+1})$ correspond to the polynomial relations $p_{n+1}(t) = (1 + t + t^2)p_n(t^2)$. These polynomials are closely related to the *Stern–Brocot sequence*. The origins of this remarkable sequence, which has attracted considerable interest in time, can be traced back to Eisenstein (see [27], [5], or the contemporary reference [26] for a thorough bibliography on this subject). In our framework the Stern–Brocot sequence $q(n, k)$, $n \geq 0$, $0 \leq k < 2^n$, simply appears as the sizes of the central summands in $\mathfrak{A}_n \cong \oplus_{k=0}^{2^{n-1}} \mathbb{M}_{q(n,k)} \oplus \mathbb{C}$, where $\mathbb{M}_r$ denotes the C^*-algebra of $r \times r$ matrices with complex entries.

The Bratteli diagram $\mathcal{G}$ has some apparent symmetries. In the last section we employ the *AF algebra path model for AF algebras* to express them, constructing sequences of projections in $\mathfrak{A}$ that satisfy certain braiding relations reminiscent of the *Temperley-Lieb-Jones relations*. In particular, for every $\tau \in (0, \frac{1}{4}]$, we construct projections $E_n \neq 0$ in $\mathfrak{A}$ such that $E_n E_{n\pm1} E_n \leq \tau E_n$ and $[E_n, E_m] = 0$ if $|n - m| \geq 2$. This suggests a possible connection with a class of statistical mechanics models with partition functions

closely related to Riemann's zeta function, called *Farey spin chains*, that have been studied in recent years by Knauf, Kleban, and their collaborators (see, e.g. [17, 18, 19, 16, 22] and references therein).

1. The Pascal triangle with memory as a Bratelli diagram

The *Pascal triangle with memory* is a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ defined as follows:

- The *vertex set* $\mathcal{V}$ is the disjoint union $\uplus_{n\geq 0}\mathcal{V}_n$ of the sets $\mathcal{V}_n = \{(n,k) : 0 \leq k \leq 2^n\}$ of *vertices at floor* n;
- The set of *edges* is defined as $\mathcal{E} = \uplus_{n\geq 0}\mathcal{E}_n$, where $\mathcal{E}_n$ is the set of edges connecting vertices at floor n with those at floor $n+1$ under the rule that (n,k) is connected with $(n+1,\ell)$ precisely when $|2k-\ell| \leq 1$. There are no edges connecting vertices from $\mathcal{V}_i$ and $\mathcal{V}_j$ when $|i-j| \geq 2$.

To each vertex (n,k) we attach the *label* $r(n,k) = \frac{p(n,k)}{q(n,k)}$, with non-negative integers $p(n,k)$, $q(n,k)$ defined recursively for $n \geq 0$ by

$$\begin{cases} q(n,0) = q(n,2^n) = 1, \quad p(n,0) = 0, \quad p(n,2^n) = 1; \\ q(n+1,2k) = q(n,k), \quad p(n+1,2k) = p(n,k), \qquad 0 \leq k \leq 2^n; \\ q(n+1,2k+1) = q(n,k) + q(n,k+1), \\ p(n+1,2k+1) = p(n,k) + p(n,k+1), \qquad 0 \leq k < 2^n. \end{cases}$$

Note that $r(n,0) = 0 < r(n,1) = \frac{1}{n+1} < \cdots < r(n,2^n) = 1$ gives a partition of $[0,1]$, and

$$p(n,k+1)q(n,k) - p(n,k)q(n,k+1) = 1, \qquad n \geq 0,\ 0 \leq k < 2^n,$$

showing in particular that $p(n,k)$ and $q(n,k)$ are relatively prime.

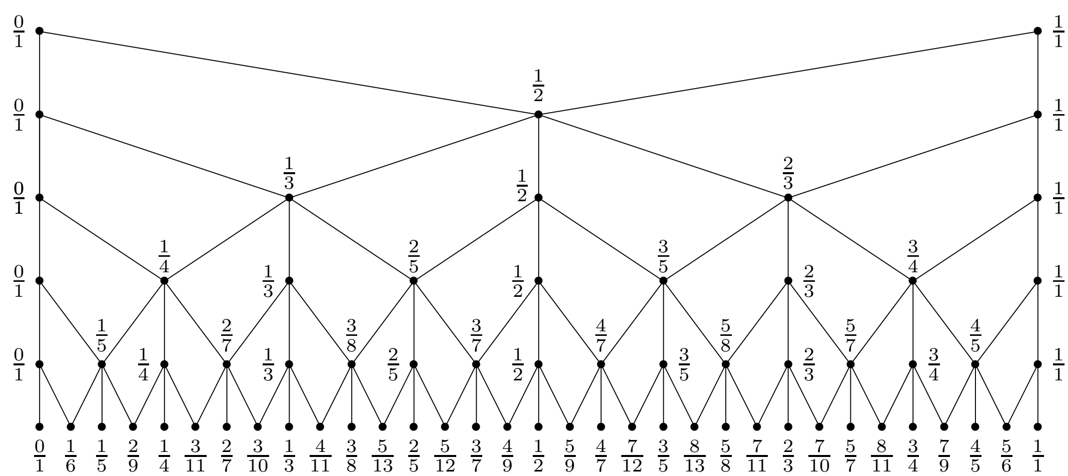


Figure 2. The Pascal triangle with memory $\mathcal{G}$

Conversely, for every pair $\frac{p}{q} < \frac{p'}{q'}$ of rational numbers with $p'q - pq' = 1$, $0 \leq p \leq q$ and $0 \leq p' \leq q'$, there exists a unique pair of integers (n,k) with $n \geq 0$, $0 \leq k < 2^n$, such that $r(n,k) = \frac{p}{q}$ and $r(n,k+1) = \frac{p'}{q'}$. This correspondence establishes a bijection between the vertices from $\mathcal{V} \setminus \{(n,2^n) : n \geq 0\}$ and the set

$$\Gamma^+ = \left\{ \begin{bmatrix} p' & p \\ q' & q \end{bmatrix} \in SL_2(\mathbb{Z}) : 0 \leq p \leq q,\ 0 \leq p' \leq q' \right\} \subset SL_2(\mathbb{Z}).$$

Remark 1. The mapping $r(n,k) \mapsto \frac{k}{2^n}$, $0 \leq k \leq 2^n$, $n \geq 0$, extends by continuity to *Minkowski's question mark function* $? : [0,1] \to [0,1]$ defined on (reduced) continued fractions as

$$?([a_1, a_2, \ldots]) \;=\; \sum_{k \geq 1} \frac{(-1)^{k-1}}{2^{(a_1+\cdots+a_k)-1}}.$$

The map ? is strictly increasing and singular, and establishes remarkable one-to-one correspondences between rational and dyadic numbers, and respectively between quadratic irrationals and rational numbers in $[0,1]$ (see [20, 7, 24]).

In this paper we shall consider the AF algebra $\mathfrak{A}$ associated with the Bratteli diagram $D(\mathfrak{A}) = \mathcal{G}$ from Figure 2. For the connection between Bratteli diagrams, AF algebras, and their ideals, we refer to the classical reference [1]. We write $(n,k) \downarrow (n',k')$ when $n' = n+1$ and there is at least one edge between the vertices (n,k) and (n',k') in the Bratteli diagram, and $(n,k) \Downarrow (n',k')$ when $n < n'$ and there are vertices $(n, k_0 = k), (n+1, k_1), \ldots, (n', k_{n'-n} = k')$ such that $(n+r, k_r) \downarrow (n+r+1, k_{r+1})$, $r = 0, \ldots, n'-n-1$. In algebraic terms this is equivalent to $e_{(n,k)} e_{(n',k')} \neq 0$, where $e_{(n,k)}$ denotes the central projection in $\mathfrak{A}_n$ that corresponds to the vertex (n,k) of the diagram. The AF algebra $\mathfrak{A}$ is the inductive limit $\varinjlim \mathfrak{A}_n$, where

$$\mathfrak{A}_n = \bigoplus_{0 \leq k \leq 2^n} \mathbb{M}_{q(n,k)}$$

and each embedding $\mathfrak{A}_n \hookrightarrow \mathfrak{A}_{n+1}$ is given by the Bratteli diagram from Figure 2.

Remark 2. Consider the set $\mathcal{V}_*$ of vertices of $\mathcal{G}$ of form (n,k) with $0 \leq k \leq 2^n$ and k odd, and the map $\Phi : \mathcal{V}_* \to \mathbb{N}$, $\Phi(n,k) = q(n,k)$. The inverse image $\Phi^{-1}(q)$ of q contains exactly $\varphi(q)$ elements, where φ denotes Euler's totient function; in particular q is prime if and only if $\#\Phi^{-1}(q) = q-1$. This remark shows, cf. [17], that the partition function associated with the corresponding Farey spin chain is $\sum_{n=1}^{\infty} \varphi(n) n^{-s}$, which is equal to $\zeta(s-1)/\zeta(s)$ when $\Re s > 2$.

Remark 3. (i) The integers $q(n,k)$ satisfy the equality

$$\sum_{0 \leq k \leq 2^n} q(n,k) = 3^n + 1.$$

(ii) Consider the Bratteli diagram obtained by deleting in $\mathcal{G}$ all vertices $(n,0)$ and denote the corresponding AF algebra by $\mathfrak{B} = \varinjlim \mathfrak{B}_n$. It is clear that $\mathfrak{B}$ is an ideal in $\mathfrak{A}$ and $\mathfrak{A}/\mathfrak{B} \cong \mathbb{C}$. Moreover,

$$\mathfrak{B}_n = \bigoplus_{1 \leq k \leq 2^n} \mathbb{M}_{p(n,k)},$$

thus the ranks of the central summands of the building blocks of $\mathfrak{B}$ give the complete list of numerators $p(n,k)$. We also have

$$\sum_{0 \leq k \leq 2^n} p(n,k) = \frac{3^n + 1}{2}.$$

2. The primitive ideal space of the AF algebra $\mathfrak{A}$

We denote $\mathbb{I} = \{\theta \in (0,1) : \theta \notin \mathbb{Q}\}$ and $\mathbb{Q}_{(0,1)} = \mathbb{Q} \cap (0,1)$.

The C^*-algebra $\mathfrak{A}$ is not simple and has a rich (and potentially interesting) structure of ideals. We first relate $\mathfrak{A}$ with the AF algebra $\mathfrak{F}_\theta$ associated by Effros and Shen [10]

to the continued fraction decomposition $\theta = [a_1, a_2, \ldots]$ of $\theta \in \mathbb{I}$. The Bratteli diagram $D(\mathfrak{F}_\theta)$ of the simple C^*-algebra $\mathfrak{F}_\theta$ is given in Figure 3.

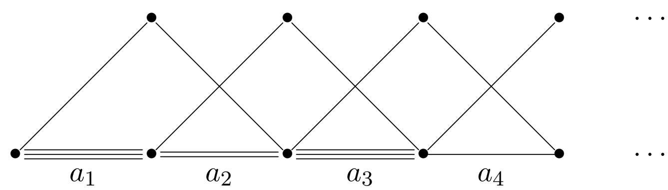


FIGURE 3. The Bratteli diagram $D(\mathfrak{F}_\theta)$

The C^*-algebra of unitized compact operators $\widetilde{\mathbb{K}} = \mathbb{C}I + \mathbb{K}$ is an AF algebra and we have a short exact sequence $0 \to \mathbb{K} \to \widetilde{\mathbb{K}} \to \mathbb{C} \to 0$, made explicit by the Bratteli diagram in Figure 4, where the shaded subdiagram corresponds to the ideal $\mathbb{K}$. Replacing $\mathbb{C} \oplus \mathbb{C}$ by $\mathbb{M}_q \oplus \mathbb{M}_{q'}$ one gets an AF algebra $\mathfrak{A}_{(q,q')}$ which is an extension of $\mathbb{K}$ by $\mathbb{M}_q$.

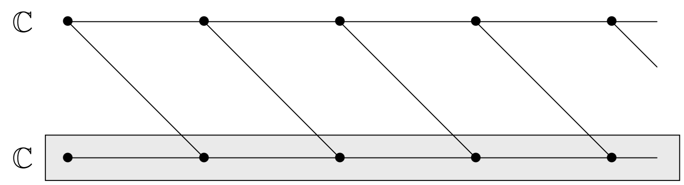


FIGURE 4. The Bratteli diagram of the C^*-algebra of unitized compact operators

We first show that Effros-Shen algebras arise naturally as quotients of our AF algebra $\mathfrak{A}$ and that the corresponding ideals belong to the primitive ideal space $\operatorname{Prim}\mathfrak{A}$. The *Farey map* $F : [0,1] \to [0,1]$ defined [14] by

$$F(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x \in \left[0, \tfrac{1}{2}\right], \\[2mm] \dfrac{1-x}{x} & \text{if } x \in \left(\tfrac{1}{2}, 1\right], \end{cases} \tag{2.1}$$

acts on infinite (reduced) continued fractions as

$$F([a_1, a_2, a_3, \ldots]) = [a_1 - 1, a_2, a_3, \ldots].$$

For each $y \in [0,1]$ the equation $F(x) = y$ has exactly two solutions $x \in [0,1]$ given by

$$x = F_1(y) = \frac{y}{1+y} \quad \text{and} \quad x = F_2(y) = \frac{1}{1+y} = 1 - F_1(y). \tag{2.2}$$

One has $F_1([a_1, a_2, \ldots]) = [a_1 + 1, a_2, \ldots]$ and $F_2([a_1, a_2, \ldots]) = [1, a_1, a_2, \ldots]$. Rational numbers are generated by the backwards orbit of F as follows:

$$\{F^{-n}(\{0\}) : n = 0, 1, 2, \ldots\} = \mathbb{Q} \cap [0,1].$$

More precisely, for each $n \in \mathbb{N}$ one has

$$\begin{aligned} F^{-n}(\{0\}) &= \{r(n-1, k) : 0 \le k \le 2^{n-1}\} \\ &= \big\{F_{i_1}^{\alpha_1} \ldots F_{i_k}^{\alpha_k}(0) : i_j \in \{1, 2\},\ i_1 \neq \cdots \neq i_k,\ \alpha_1 + \cdots + \alpha_k = n\big\} \\ &= \big\{[a_1, \ldots, a_r] : a_1 + \cdots + a_r \le n\big\}. \end{aligned}$$

In the next statement, given relatively prime integers $0 < p < q$, $\overline{p}$ will denote the multiplicative inverse of p modulo q in $\{1, \dots, q-1\}$.

Proposition 4. (i) *For each $\theta \in \mathbb{I}$, there is $I_\theta \in \operatorname{Prim}\mathfrak{A}$ such that $\mathfrak{A}/I_\theta \cong \mathfrak{F}_\theta$.*

(ii) *Given $\theta = \frac{p}{q} \in \mathbb{Q}_{(0,1)}$ in lowest terms, there are $I_\theta, I_\theta^+, I_\theta^- \in \operatorname{Prim}\mathfrak{A}$ such that $\mathfrak{A}/I_\theta \cong \mathbb{M}_q$, $\mathfrak{A}/I_\theta^- \cong \mathfrak{A}_{(q,\overline{p})}$, and $\mathfrak{A}/I_\theta^+ \cong \mathfrak{A}_{(q,q-\overline{p})}$.*

(iii) *There are $I_0, I_0^+, I_1, I_1^- \in \operatorname{Prim}\mathfrak{A}$ such that $\mathfrak{A}/I_0 \cong \mathfrak{A}/I_1 \cong \mathbb{C}$ and $\mathfrak{A}/I_0^+ \cong \mathfrak{A}/I_1^- \cong \widetilde{\mathbb{K}}$.*

Proof. (i) Let $\theta \in \mathbb{I}$ with continued fraction $[a_1, a_2, \dots]$ and $r_\ell = r_\ell(\theta) = p_\ell/q_\ell = [a_1, \dots, a_\ell]$ be its ℓ^{th} convergent, where $p_\ell = p_\ell(\theta)$ and $q_\ell = q_\ell(\theta)$ can be recursively defined by

$$\begin{cases} p_{-1} = 1,\ q_{-1} = 0, \quad p_0 = 0,\ q_0 = 1; \\ \begin{bmatrix} p_\ell & q_\ell \\ p_{\ell-1} & q_{\ell-1} \end{bmatrix} = \begin{bmatrix} a_\ell & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{\ell-1} & q_{\ell-1} \\ p_{\ell-2} & q_{\ell-2} \end{bmatrix}, \qquad \ell \geq 1. \end{cases}$$

The relation $p_\ell q_{\ell-1} - p_{\ell-1} q_\ell = (-1)^{\ell-1}$ shows in particular that $\gcd(p_\ell, q_\ell) = 1$.

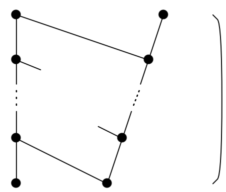
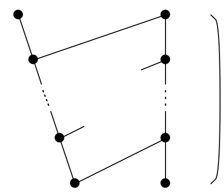
$L_a =$  a spaces $\qquad R_a =$  a spaces

FIGURE 5. The diagrams L_a and R_a

For each $a \in \mathbb{N} = \{1, 2, \dots\}$ consider the diagrams L_a and R_a from Figure 5. Also set $L_0 = R_0 = \emptyset$. Clearly L_{a+b} coincides with the concatenation $L_a \circ L_b$ of L_a followed by L_b, and we also have $R_{a+b} = R_a \circ R_b$. Using the obvious identifications between $L_a \circ R_b$, $R_a \circ L_b$ and $C_a \circ C_b$ (see Figure 6), and (2.2), we see that the AF algebras generated by $L_{a_1} \circ R_{a_2} \circ L_{a_3} \circ R_{a_4} \circ \dots$ and $R_{a_1} \circ L_{a_2} \circ R_{a_3} \circ L_{a_4} \circ \dots$ are isomorphic to $\mathfrak{F}_{[a_1+1, a_2, a_3, \dots]} \simeq \mathfrak{F}_{F_1(\theta)} \simeq \mathfrak{F}_{F_2(\theta)} \simeq \mathfrak{F}_{[1, a_1, a_2, \dots]}$ (note that the AF algebra defined by $C_{a_1} \circ C_{a_2} \circ C_{a_3} \circ \cdots$ is isomorphic to $\mathfrak{F}_{[a_1+1, a_2, a_3, \dots]}$).

The Bratteli subdiagram $\mathcal{G}_\theta$ of $\mathcal{G}$ containing the vertices $(0,0)$ and $(0,1)$ and defined by $L_{a_1-1} \circ R_{a_2} \circ L_{a_3} \circ R_{a_4} \circ \cdots$ generates a copy of $\mathfrak{F}_\theta$. The complement $\mathcal{G} \setminus \mathcal{G}_\theta$ is a directed and hereditary Bratteli diagram as in [1, Lemma 3.2] (see also Figure 7). Thus there is an ideal I_θ in $\mathfrak{A}$ such that $D(I_\theta) = \mathcal{G} \setminus \mathcal{G}_\theta$, $D(\mathfrak{A}/I_\theta) = \mathcal{G}_\theta$, and $\mathfrak{A}/I_\theta \cong \mathfrak{F}_\theta$. Moreover I_θ is a primitive ideal cf. [1, Theorem 3.8].

If $j_n = j_n(\theta)$ is the unique index for which $r(n, j_n) < \theta < r(n, j_n + 1)$ (see Figure 7), then

$$I_\theta \cap \mathfrak{A}_n = \bigoplus_{\substack{0 \leq k \leq 2^n \\ k \neq j_n, j_{n+1}}} \mathbb{M}_{q(n,k)}.$$

The vertices of $D(\mathfrak{A}/I_\theta)$ are explicitly related to the continued fraction decomposition of θ. For each $r \in \mathbb{Q}_{(0,1)}$, denote $\operatorname{ht}(r) = \min\{n : \exists k,\ r(n,k) = r\}$. Let $\frac{p_n}{q_n}$ be the continued fraction approximations of θ, and $h_n = \operatorname{ht}(\frac{p_n}{q_n})$. With this notation, the labels of the two vertices at floor m in $\mathcal{G}_\theta$ are $\frac{p_n}{q_n}$ and $\frac{p_{n-1} + (m-h_n)p_n}{q_{n-1} + (m-h_n)q_n}$ whenever $h_n \leq m < h_{n+1}$.

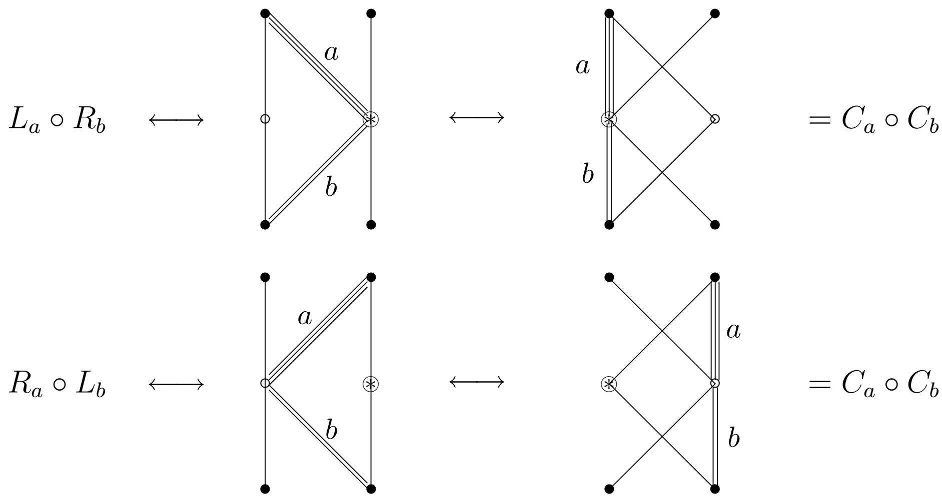


FIGURE 6. The identification between $L_a \circ R_b$, $R_a \circ L_b$, and $C_a \circ C_b$

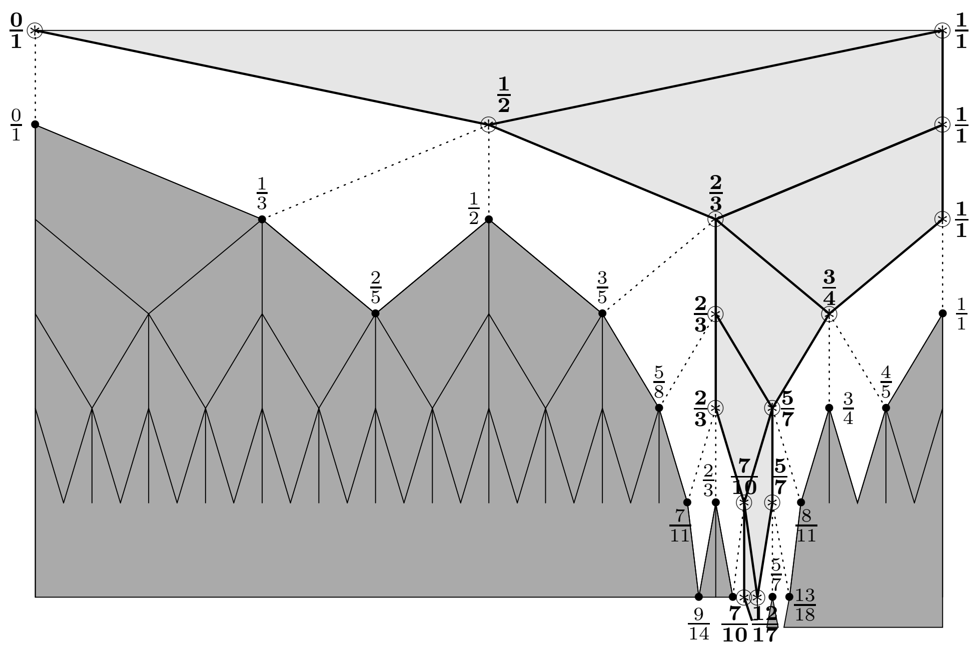


FIGURE 7. The diagrams $\mathcal{G}_\theta = D(\mathfrak{A}/I_\theta) = R_2 \circ L_2 \circ R_1 \circ L_1 \circ \cdots$ (lighter) and $\mathcal{G} \setminus \mathcal{G}_\theta = D(I_\theta)$ (darker) when $\theta = [1, 2, 2, 1, 1, \dots]$

(ii) For each $\theta = \frac{p}{q} \in \mathbb{Q}_{(0,1)}$ in lowest terms, consider the Bratteli subdiagram $\mathcal{G}_\theta$ of $\mathcal{G}$ defined by all vertices (n, j) with $r(n, j) = \theta$ and (m, i) with $(m, i) \Downarrow (n, j)$. The AF algebra associated to $\mathcal{G}_\theta$ is clearly isomorphic to $\mathbb{M}_q$. Again, the complement $\mathcal{G} \setminus \mathcal{G}_\theta$ is seen to be a directed and hereditary Bratteli diagram. Therefore there is a primitive ideal I_θ in $\mathfrak{A}$ such that $D(I_\theta) = \mathcal{G} \setminus \mathcal{G}_\theta$ and $\mathfrak{A}/I_\theta \simeq \mathbb{M}_q$.

Let $n_0 - 1 = n_0(\theta) - 1$ be the largest $n \in \mathbb{N}$ for which there exists $j = j_n(\theta)$ such that $r(n, j) < \theta < r(n, j + 1)$. For $n < n_0$ define j_n as above. By the choice of n_0 and the properties of the Pascal triangle with repetition, for every $n \geq n_0$ there is $j_n = j_n(\theta)$

with $r(n,j_n)=\theta$. The ideal I_θ is generated by the direct summands $\mathbb{M}_{q(n_0,j_{n_0}-1)}$, $\mathbb{M}_{q(n_0,j_{n_0}+1)}$ and $\mathbb{M}_{q(n,c_n)}$, $n<n_0$, that is

$$I_\theta \cap \mathfrak{A}_n = \begin{cases} \bigoplus\limits_{\substack{0\leq k\leq 2^n \\ k\neq j_n, j_{n+1}}} \mathbb{M}_{q(n,k)} & \text{if } n<n_0, \\ \bigoplus\limits_{\substack{0\leq k\leq 2^n \\ k\neq j_n}} \mathbb{M}_{q(n,k)} & \text{if } n\geq n_0. \end{cases}$$

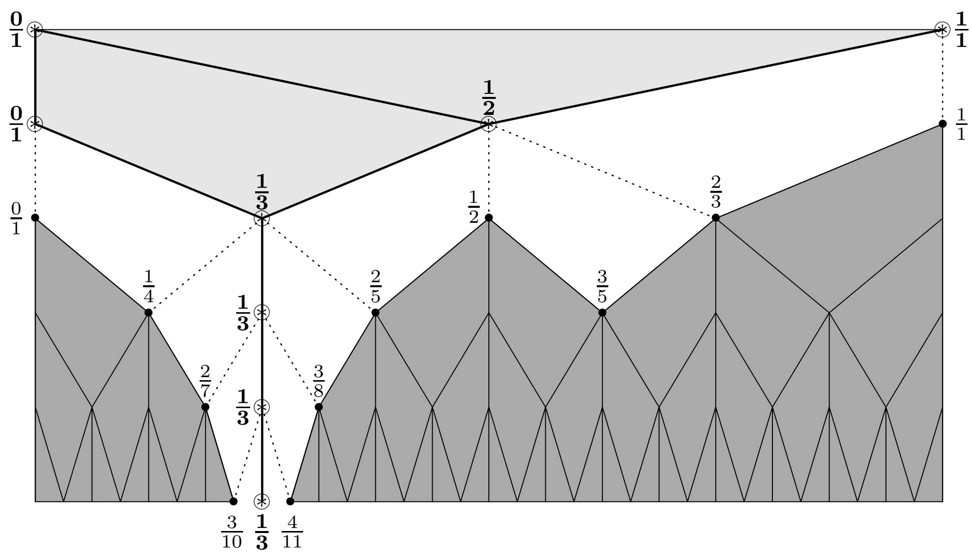


FIGURE 8. The diagrams $D(I_{\frac{1}{3}})$ (darker) and $D(\mathfrak{A}/I_{\frac{1}{3}})$ (lighter)

The ideals $I_\theta^{\pm}$ defined by (see also Figures 9 and 10)

$$I_\theta^{+} \cap \mathfrak{A}_n = \bigoplus_{\substack{0\leq k\leq 2^n \\ k\neq j_n, j_{n+1}}} \mathbb{M}_{q(n,k)},$$

and respectively by

$$I_\theta^{-} \cap \mathfrak{A}_n = \begin{cases} \bigoplus\limits_{\substack{0\leq k\leq 2^n \\ k\neq j_n, j_{n+1}}} \mathbb{M}_{q(n,k)} & \text{if } n<n_0, \\ \bigoplus\limits_{\substack{0\leq k\leq 2^n \\ k\neq j_{n-1}, j_n}} \mathbb{M}_{q(n,k)} & \text{if } n\geq n_0, \end{cases}$$

are primitive and we clearly have $\mathfrak{A}/I_\theta^{-} \cong \mathfrak{A}_{(q,\overline{p})}$ and $\mathfrak{A}/I_\theta^{+} \cong \mathfrak{A}_{(q,q-\overline{p})}$.

(iii) is now obvious. $\square$

Remark 5. A joint (and important) feature of all cases above is that

$$(n,j)\notin D(I_\theta)=\mathcal{G}\setminus\mathcal{G}_\theta \implies r(n,j-1)<\theta<r(n,j+1).$$

Remark 6. In $GL_2(\mathbb{Z})$ consider the matrices

$$A=\begin{bmatrix}1&0\\1&1\end{bmatrix},\quad B=\begin{bmatrix}1&1\\0&1\end{bmatrix},\quad J=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad M(a)=\begin{bmatrix}a&1\\1&0\end{bmatrix}.$$

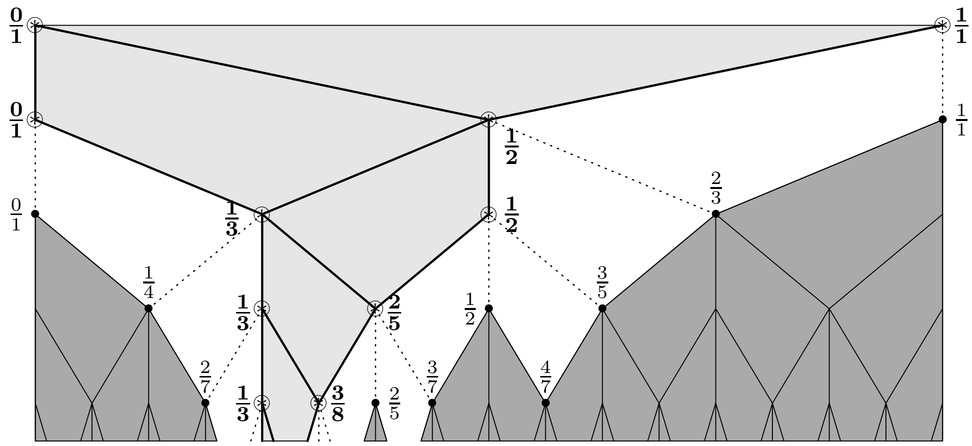


FIGURE 9. The diagrams $D(I^+_{\frac{1}{3}})$ (darker) and $D(\mathfrak{A}/I^+_{\frac{1}{3}})$ (lighter)

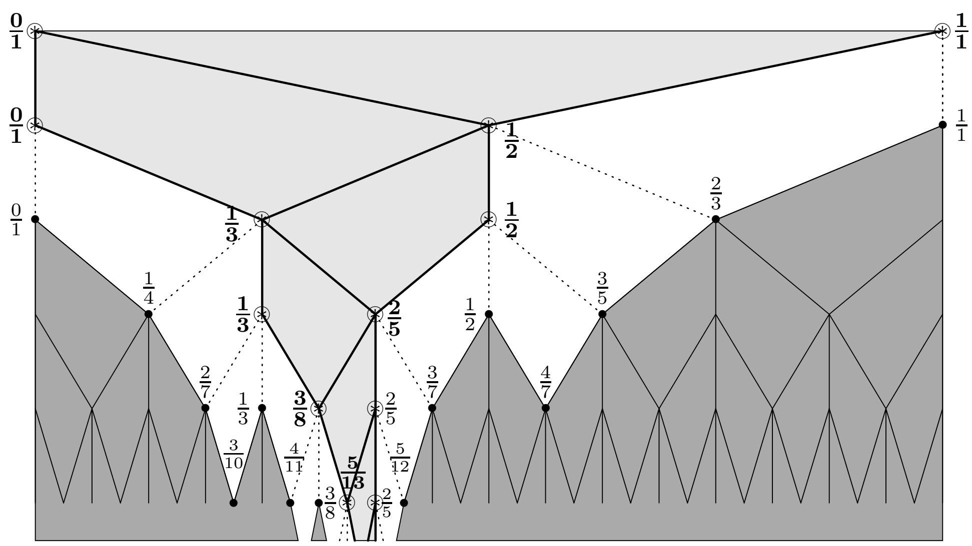


FIGURE 10. The diagrams $D(I^-_{\frac{2}{5}})$ (darker) and $D(\mathfrak{A}/I^-_{\frac{2}{5}})$ (lighter)

The identification between $L_a \circ R_b$ and $C_a \circ C_b$ reflects the matrix equality

$$B^a A^b = M(a)M(b),$$

whereas the identification between $R_a \circ R_b$ and $C_a \circ C_b$ reflects the matrix equality

$$A^a B^b = JM(a)M(b)J.$$

A combinatorial analysis based on Bratteli's correspondence between primitive ideals and subdiagrams of $\mathcal{G}$ shows that these are actually the only primitive ideals of $\mathfrak{A}$.

Proposition 7. $\operatorname{Prim}\mathfrak{A} = \{I_\theta : \theta \in \mathbb{I}\} \cup \{I_\theta, I_\theta^\pm : \theta \in \mathbb{Q}_{(0,1)}\} \cup \{I_0, I_0^+, I_1, I_1^-\}$.

Proof. Let $I \in \operatorname{Prim}\mathfrak{A}$. Consider the Bratteli diagrams $D = D(I)$ and $\widetilde{D} = D(\mathfrak{A}/I) = \mathcal{G} \setminus D$. If there is n_0 such that $(n_0, k) \in D$ for all $0 \le k \le 2^{n_0}$, then $I = \mathfrak{A}$. So for each n the set $L_n = \{k : (n,k) \in \widetilde{D}\}$ is nonempty. Denote also $L_n^c = \{0, 1, \ldots, 2^n\} \setminus L_n$.

We first notice that L_n should be a set of the form $\{a_n\}$ or $\{a_n, a_n+1\}$. If not, there are $k, k' \in L_n$ such that $k'-k \geq 2$. Since I is a primitive ideal, a vertex (p,r) in $\mathcal{G}$ should exist such that $(n,k) \Downarrow (p,r)$ and $(n,k') \Downarrow (p,r)$. Since $k'-k > 2$ this is not possible due to the definition of $\mathcal{G}$.

To finish the proof it suffices to show that

$$L_{n+1} = \begin{cases} \{2a_n\} & \text{if } L_n = \{a_n\}, \\ \{2a_n, 2a_n+1\},\ \{2a_n+1, 2a_n+2\}, & \\ \quad \text{or } \ \{2a_n+1\} & \text{if } L_n = \{a_n, a_n+1\}, \end{cases} \tag{2.3}$$

that is, all links $(n,j) \downarrow (n+1,j')$ in $\widetilde{D}$ are exactly as indicated in Figure 11.

Indeed, if $L_n = \{a_n\}$, then $(n, a_n-1), (n, a_n+1)$ are vertices in the hereditary diagram D; thus we also have $(n+1, 2a_n-1), (n+1, 2a_n+1) \in D$. Because D is directed, $(n+1, 2a_n) \in D$ would imply $(n, a_n) \in D$, which contradicts $a_n \in L_n$.

If $L_n = \{a_n, a_n+1\}$, then $(n, a_n-1), (n, a_n+2) \in D$. Moreover because D is hereditary the vertices $(n+1, 2a_n-1)$ and $(n+1, 2a_n+3)$ also belong to D. We now look at the consecutive vertices $(n+1, 2a_n), (n+1, 2a_n+1), (n+1, 2a_n+2)$. From the first part they cannot all belong to $\widetilde{D}$. If $(n+1, 2a_n+1) \in D$, and $(n+1, 2a_n), (n+1, 2a_n+2) \in \widetilde{D}$, then L_{n+1} has a gap, thus contradicting the first part. If $(n+1, 2a_n), (n+1, 2a_n+2) \in D$ it follows, as a result of the fact that $(n+1, 2a_n-1) \in D$ and that D is directed, that $(n+1, 2a_n+1) \in \widetilde{D}$. In a similar way one cannot have $(n+1, 2a_n+1), (n+1, 2a_n+2) \in D$. It remains that only the following cases can occur (see also Figure 11):

(i) $(n+1, 2a_n), (n+1, 2a_n+1) \in \widetilde{D}$ and $(n+1, 2a_n+2) \in D$, thus $L_{n+1} = \{2a_n, 2a_n+1\}$.

(ii) $(n+1, 2a_n) \in D$ and $(n+1, 2a_n+1), (n+1, 2a_n+2) \in \widetilde{D}$, thus $L_{n+1} = \{2a_n+1, 2a_n+2\}$.

(iii) $(n+1, 2a_n+1) \in \widetilde{D}$ and $(n+1, 2a_n), (n+1, 2a_n+2) \in D$, thus $L_{n+1} = \{2a_n+1\}$,

which concludes the proof of (2.3). $\square$

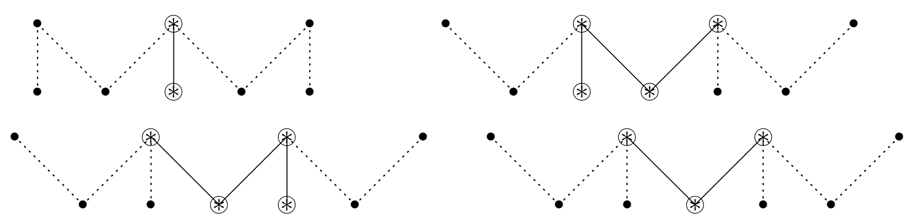


FIGURE 11. The possible links between two consecutive floors in $D(\mathfrak{A}/I)$

3. THE JACOBSON TOPOLOGY ON $\operatorname{Prim}\mathfrak{A}$

We first recall some basic things about the primitive ideal space of a C^*-algebra $\mathcal{A}$ following [8] and [23]. For each set $S \subseteq \operatorname{Prim}\mathcal{A}$, consider the ideal $k(S) := \cap_{J \in S} J$ in $\mathcal{A}$, called the *kernel* of S. For each ideal I consider its *hull*, $h(I) := \{P \in \operatorname{Prim}\mathcal{A} : I \subseteq P\}$. The *closure* of a set $S \subseteq \operatorname{Prim}\mathcal{A}$ is defined as

$$\overline{S} := \{P \in \operatorname{Prim}\mathcal{A} : k(S) \subseteq P\}.$$

There is a unique topology on $\operatorname{Prim}\mathcal{A}$, called the *Jacobson* (or *hull-kernel*) *topology* such that its closed sets are exactly those with $S = \overline{S}$. The open sets in $\operatorname{Prim}\mathcal{A}$ are then precisely those of the form

$$\mathcal{O}_I := \{P \in \operatorname{Prim}\mathcal{A} : I \nsubseteq P\}$$

for some ideal I in $\mathcal{A}$. The Jacobson topology is always T_0, i.e. for any two distinct points in $\operatorname{Prim}\mathcal{A}$ one of them has a neighborhood which does not contain the other.

Moreover, the correspondence $S \mapsto k(S)$ establishes a one-to-one correspondence between the closed subsets S of $\operatorname{Prim}\mathcal{A}$ and the lattice of ideals in $\mathcal{A}$, with inverse given by $I \mapsto h(I)$. For any ideal I in $\mathcal{A}$, let p_I denote the quotient map $\mathcal{A} \to \mathcal{A}/I$. The mapping $P \mapsto P \cap I$ is a homeomorphism of the open set $\mathcal{O}_I$ onto $\operatorname{Prim} I$, whereas $Q \mapsto p_I^{-1}(Q)$ is a homeomorphism of $\operatorname{Prim}\mathcal{A}/I$ onto the closed set $h(I)$ of $\operatorname{Prim}\mathcal{A}$. A general study of the primitive ideal space of AF algebras was pursued in [2, 4, 9].

We collect some immediate properties of the primitive ideal space of $\mathfrak{A}$ in the following

Remark 8. (i) For each $\theta \in \mathbb{I}$, $\overline{\{I_\theta\}} = \{I_\theta\}$.

(ii) For each $\theta \in \mathbb{Q}_{(0,1)}$, $I_\theta \nsubseteq I_\theta^+$, $I_\theta \nsubseteq I_\theta^-$, and $I_\theta = I_\theta^+ \cap I_\theta^-$. We also have $I_0 \nsubseteq I_0^+$ and $I_1 \nsubseteq I_1^-$. Therefore $\overline{\{I_\theta\}} = \{I_\theta, I_\theta^+, I_\theta^-\}$ whenever $\theta \in \mathbb{Q}_{(0,1)}$, $\overline{\{I_0\}} = \{I_0, I_0^+\}$ and $\overline{\{I_1\}} = \{I_1, I_1^-\}$, showing in particular that the Jacobson topology on $\operatorname{Prim}\mathfrak{A}$ is not Hausdorff. In spite of this we shall see that after removing the "singular points" $I_\theta^\pm$ from $\operatorname{Prim}\mathfrak{A}$ we retrieve the usual topology on $[0,1]$.

For each set $E \subseteq [0,1]$, consider the ideal

$$\mathfrak{I}(E) := \bigcap_{\theta \in E} I_\theta, \tag{3.1}$$

and denote by $\overline{E}$ the usual closure of E in $[0,1]$.

Lemma 9. $\mathfrak{I}(E) = \mathfrak{I}(\overline{E})$ *for every set* $E \subseteq [0,1]$.

Proof. The inclusion $\mathfrak{I}(\overline{E}) \subseteq \mathfrak{I}(E)$ is obvious by (3.1). We prove $\mathfrak{I}(E) \subseteq I_x$ for all $x \in \overline{E}$. Suppose ad absurdum there is $x \in \overline{E}$ for which $\mathfrak{I}(E) \nsubseteq I_x$, i.e. there is $(n,j) \in \mathcal{V}$ with $(n,j) \in D(\mathfrak{I}(E))$ and $(n,j) \notin D(I_x)$. The latter and Remark 5 yield

$$r(n, j-1) < x < r(n, j+1). \tag{3.2}$$

On the other hand, because $D(\mathfrak{I}(E))$ contains (n,j), every diagram $D(I_\theta)$, $\theta \in E$, must contain the whole "pyramid" starting at (n,j), see Figure 12. Thus

$$\forall \theta \in E,\ \forall k \geq 1, \quad \theta \in [0, r(n+k, 2^k j - 2^k + 1), 1] \cup [r(n+k, 2^k j + 2^k - 1), 1].$$

But

$$r(n+k, 2^k j + 2^k - 1) = \frac{kp(n,j+1) + p(n,j)}{kq(n,j+1) + q(n,j)} \xrightarrow{k} \frac{p(n,j+1)}{q(n,j+1)} = r(n, j+1)$$

and

$$r(n+k, 2^k j - 2^k + 1) = \frac{kp(n,j-1) + p(n,j)}{kq(n,j-1) + q(n,j)} \xrightarrow{k} \frac{p(n,j-1)}{q(n,j-1)} = r(n, j-1),$$

hence

$$E \subseteq [0, r(n, j-1)] \cup [r(n, j+1), 1],$$

which is in contradiction with (3.2). $\square$

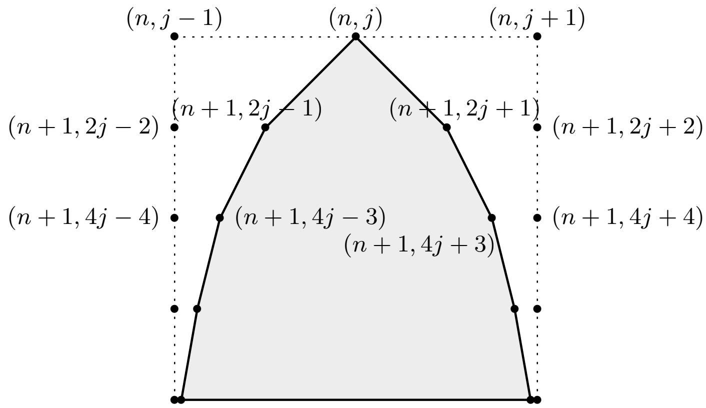


FIGURE 12. The ideal generated by (n, j)

Remark 10. We have $q(n, 2j) = q(n-1, j) < \min\{q(n, 2j-1), q(n, 2j+1)\}$, so if $r(n, 2j) = \frac{p}{q}$, then

$$r(n, 2j+1) - r(n, 2j-1) = \frac{1}{q(n, 2j-1)q(n, 2j)} + \frac{1}{q(n, 2j)q(n, 2j+1)} < \frac{2}{q^2}.$$

One can give a better estimate as follows. Let $\theta = \frac{p}{q} \in (0, 1)$ be a rational number in lowest terms and let $\bar{p} \in \{1, \ldots, q-1\}$ denote the multiplicative inverse of p modulo q. Let $n_0 = n_0(\theta)$ be the smallest n such that $\theta = r(n, j_0)$ for some j_0. Then j_0 is odd and the labels $r' = \frac{p'}{q'}$ and respectively $r'' = \frac{p''}{q''}$ of the "left parent" $(n_0 - 1, \frac{j_0-1}{2})$ and respectively of the "right parent" $(n_0 - 1, \frac{j_0+1}{2})$ of the vertex (n_0, j_0), are given by $(q', p') = \big(\bar{p}, \frac{p\bar{p}-1}{q}\big)$, and respectively by $(q'', p'') = \big(q - \bar{p}, p - \frac{p\bar{p}-1}{q}\big) = (q, p) - (q', p')$. Furthermore, we have $r(n_0 + k, 2^k j_0 - 1) = \frac{p+kp'}{q+kq'}$, $r(n_0 + k, 2^k j_0 + 1) = \frac{p+kp''}{q+kq''}$, and

$$\max\left\{ r(n_0 + k, 2^k j_0 + 1) - \frac{p}{q} \ , \ \frac{p}{q} - r(n_0 + k, 2^k j_0 - 1) \right\} < \frac{1}{kq^2}.$$

Lemma 11. *For some $x \in [0, 1]$ and $S \subseteq [0, 1]$ suppose $\mathfrak{I}(S) \subseteq \mathfrak{I}_x$. Then $x \in \overline{S}$.*

Proof. Obviously two cases may occur:

Case I: $x \notin \mathbb{Q}$. Let $\big(\frac{p_n}{q_n}\big)$ denote the sequence of continued fraction approximations of x. Taking stock on the definition of the ideal $\mathfrak{I}_x$ we get positive integers $k_1 < k_2 < \cdots$ and vertices $(k_n, j_n) \in D(\mathfrak{A})$ with the following properties:

(i) $r(k_n, j_n) = \dfrac{p_n}{q_n}$;
(ii) j_n is even;
(iii) $(k_n, j_n) \notin D(\mathfrak{I}_x)$.

Actually (iii) is a plain consequence of (i) and gives in turn, cf. Remark 5,

$$r(k_n, j_n - 1) < x < r(k_n, j_n + 1). \tag{3.3}$$

Case II: $x \in \mathbb{Q}$. There is n_0 such that $(n, j_n) \notin D(\mathfrak{I}_x)$ and $r(n, j_n) = x$ for all $n \geq n_0$. In this case we take $k_n = n$.

Suppose that $\exists n \geq n_0$, $\forall \theta \in S$, $(k_n, j_n) \in D(\mathfrak{I}_\theta)$. Then $(k_n, j_n) \in D(\mathfrak{I}(S)) \setminus D(\mathfrak{I}_x)$, which contradicts the assumption of the lemma. Therefore we must have

$$\forall n,\ \exists \theta_n \in S,\ (k_n, j_n) \notin D(\mathfrak{I}_{\theta_n}),$$

which according to Remark 5 gives

$$r(k_n, j_n - 1) < \theta_n < r(k_n, j_n + 1). \tag{3.4}$$

From (3.3), (3.4) and Remark 10 we now infer

$$|x - \theta_n| < r(k_n, j_n + 1) - r(k_n, j_n - 1) < \frac{2}{q_n^2}, \qquad \forall n \geq n_0,$$

and so $\operatorname{dist}(x, S) = 0$. This concludes the proof of the lemma. □

As a consequence, the Jacobson topology is Hausdorff when restricted to the subset $\operatorname{Prim}_0 \mathfrak{A} = \{I_\theta : \theta \in [0,1]\}$ of $\operatorname{Prim} \mathfrak{A}$. Moreover, we have

Corollary 12. *Let (θ_n) be a sequence in $[0,1]$. The following are equivalent:*
- (i) $\theta_n \to \theta$ *in* $[0,1]$.
- (ii) $I_{\theta_n} \to I_\theta$ *in* $\operatorname{Prim} \mathfrak{A}$.

Proof. (i) Suppose $\theta_n \to \theta$ in $[0,1]$ but $I_{\theta_n} \nrightarrow I_\theta$ in $\operatorname{Prim} \mathfrak{A}$. Then there is I ideal in $\mathfrak{A}$ such that $I \nsubseteq I_\theta$ and there is a subsequence (n_k) such that $I_{\theta_{n_k}} \notin \mathcal{O}_I$, so that $I \subseteq I_{\theta_{n_k}}$. By Lemma 9 this also yields $I \subseteq I_\theta$, which is a contradiction.

(ii) Suppose $I_{\theta_n} \to I_\theta$ in $\operatorname{Prim} \mathfrak{A}$ but $\theta_n \nrightarrow \theta$ in $[0,1]$. Then there is a subsequence (n_k) such that $\theta \notin \overline{\{\theta_{n_k}\}}$. By Lemma 11 we have $I := \cap_k I_{\theta_{n_k}} \nsubseteq I_\theta$, and so $I_\theta \in \mathcal{O}_I$. But on the other hand $I \subseteq I_{\theta_{n_k}}$, i.e. $I_{\theta_{n_k}} \notin \mathcal{O}_I$ for all k, thus contradicting $I_{\theta_{n_k}} \to I_\theta$. □

4. A description of the dimension group

By a classical result of Elliott ([12], see also [11]), AF algebras are classified up to isomorphism by their dimension groups. In this section we give a description of the dimension group $K_0(\mathfrak{C})$ of the codimension one ideal $\mathfrak{C} = I_1$ of $\mathfrak{A}$ obtained by erasing all vertices $(n, 2^n)$ from the Bratteli diagram. This is inspired by the generating function identity [6]

$$\sum_{n \geq 0} \theta_n X^n = \prod_{k \geq 0} (1 + X^{2^k} + X^{2^{k+1}}),$$

where (θ_n) is the Stern-Brocot sequence $q(0,0)$, $q(1,0)$, $q(1,1)$, $q(2,0)$, $q(2,1)$, $q(2,2)$, $q(2,3)$, …, $q(n,0)$, …, $q(n, 2^n - 1)$, $q(n+1, 0)$, …

For each integer $n \geq 0$, set

$$p_{(n,k)}(X) := \begin{cases} 1 & \text{if } k = 0, \\ X^k + X^{-k} & \text{if } 1 \leq k < 2^n, \end{cases}$$

and consider the abelian additive group

$$\mathcal{P}_n := \Big\{ \sum_{0 \leq k < 2^n} c_k p_{(n,k)} : c_k \in \mathbb{Z} \Big\}.$$

Set

$$\varrho(X) = X^{-1} + 1 + X, \quad \varrho_n(X) = \prod_{0 \leq k < n} \varrho(X^{2^k}),$$

and define the injective group morphisms

$$\beta_m : \mathcal{P}_m \to \mathcal{P}_{m+1}, \quad (\beta_m(p))(X) = \varrho(X)p(X^2),$$

$$\beta_{m,n} : \mathcal{P}_m \to \mathcal{P}_n, \quad (\beta_{m,n}(p))(X) = (\beta_{n-1}\cdots\beta_m(p))(X) = \varrho_{m-n}(X)p(X^{2^{n-m}}), \quad m < n.$$

Note that

$$\begin{aligned}(\beta_n(p_{(n,k)}))(X) &= \varrho(X)p_{(n,k)}(X^2)\\ &= \begin{cases} p_{(n+1,0)}(X) + p_{(n+1,1)}(X) & \text{if } k = 0,\\ p_{(n+1,2k-1)}(X) + p_{(n+1,2k)}(X) + p_{(n+1,2k+1)}(X) & \text{if } 1 \le k < 2^n.\end{cases}\end{aligned} \tag{4.1}$$

The group $K_0(\mathfrak{C}_n)$ identifies with the free abelian group $\mathbb{Z}^{2^n}$, generated by the Murray-von Neumann equivalence classes $[e_{(n,k)}]$ of minimal projections $e_{(n,k)}$ in the central summand $\mathfrak{A}_{(n,k)}$, $0 \le k < 2^n$. We have $K_0(\mathfrak{C}) = \varinjlim(K_0(\mathfrak{C}_n), \alpha_n)$, the injective morphisms $\alpha_n : K_0(\mathfrak{C}_n) \to K_0(\mathfrak{C}_{n+1})$ being given by

$$\alpha_n([e_{(n,k)}]) = \begin{cases} [e_{(n+1,0)}] + [e_{(n+1,1)}] & \text{if } k = 0,\\ [e_{(n+1,2k-1)}] + [e_{(n+1,2k)}] + [e_{(n+1,2k+1)}] & \text{if } 1 \le k < 2^n.\end{cases}$$

The positive cone $K_0(\mathfrak{C}_n)^+$ consists of elements of form $\sum_{k=0}^{2^n-1} c_k[e_{(n,k)}]$, $c_k \in \mathbb{Z}_+$. The groups $K_0(\mathfrak{C}_n)$ and $\mathcal{P}_n$ are identified by the group isomorphism ϕ_n mapping $[e_{(n,k)}]$ onto $p_{(n,k)}$. Equalities (4.1) are reflected into the commutativity of the diagram

$$\begin{tikzcd} K_0(\mathfrak{C}_n) \arrow[r, "\phi_n"] \arrow[d, "\alpha_n"'] & \mathcal{P}_n \arrow[d, "\beta_n"] \\ K_0(\mathfrak{C}_{n+1}) \arrow[r, "\phi_{n+1}"'] & \mathcal{P}_{n+1} \end{tikzcd} \tag{4.2}$$

As a result, $K_0(\mathfrak{C})$ is isomorphic with the abelian group $\mathcal{P} = \varinjlim(\mathcal{P}_n, \beta_n)$ and can, therefore, be described as $(\cup_n \mathcal{P}_n)/_\sim = \mathbb{Z}[X + X^{-1}]/_\sim$ where $\sim$ is the equivalence relation given by equality on each $\mathcal{P}_n \times \mathcal{P}_n$, and for $p \in \mathcal{P}_m$, $q \in \mathcal{P}_n$, $m < n$, by

$$p \sim q \iff q(X) = (\beta_{m,n}(p))(X) = p(X^{2^{n-m}}) \prod_{0 \le k < n-m} (X^{-2^k} + 1 + X^{2^k}).$$

Let $[p]$ denote the equivalence class of $p \in \cup_n \mathcal{P}_n$. The addition on $\mathcal{P}$ is given by

$$[p] + [q] = [\beta_{m,n}(p) + q], \quad p \in \mathcal{P}_m,\ q \in \mathcal{P}_n,\ m \le n,$$

and does not depend on the choice of m or n. For example

$$\begin{aligned}[X^{-1} + X] + [X^{-3} + X^3] &= [(X^{-1} + 1 + X)(X^{-2} + X^2) + X^{-3} + X^3]\\ &= [2(X^{-3} + X^3) + (X^{-2} + X^2) + (X^{-1} + X)].\end{aligned}$$

An element $[p]$, $p \in \mathcal{P}_n$, belongs to the positive cone $\mathcal{P}^+$ of the dimension group precisely when there is an integer $N > n$ such that $\beta_{n,N}(p)$ has nonnegative coefficients. The equality (where $c_{r+1} = 0$)

$$\begin{aligned}&(X^{-1} + 1 + X) \sum_{0 \le k < 2^n} c_k(X^{2k} + X^{-2k})\\ &\qquad = \sum_{0 \le k < 2^n} c_k(X^{2k} + X^{-2k}) + \sum_{0 \le k < 2^n} (c_k + c_{k+1})(X^{2k+1} + X^{-2k-1})\end{aligned}$$

shows that $p(X)$ has nonnegative coefficients if and only if $\varrho(X)p(X^2)$ has the same property. Therefore $[p] \in \mathcal{P}^+$ precisely when $p(X)$ has nonnegative coefficients.

Consider the positive integers $q'_{(n,k)}$, $n \geq 0$, $0 \leq k < 2^n$, describing the sizes of central summands in

$$\mathfrak{C}_n = \bigoplus_{0 \leq k < 2^n} \mathbb{M}_{q'_{(n,k)}}, \tag{4.3}$$

that is

$$\begin{cases} q'_{(n,0)} = q'_{(n,2^n-1)} = 1, \\ q'_{(n,2k)} = q'_{(n-1,k)}, \\ q'_{(n,2k+1)} = q'_{(n-1,k)} + q'_{(n-1,k+1)}, \quad 0 \leq k < 2^n. \end{cases}$$

For instance $q'(3,k)$, $0 \leq k \leq 7$, are given by $1,3,2,3,1,2,1,1$, and $q'(4,k)$, $0 \leq k \leq 15$, by $1,4,3,5,2,5,3,4,1,3,2,3,1,2,1,1$. From (4.3) we have

$$\sum_{0 \leq k < 2^n} q'(n,k)[e_{(n,k)}] = [1] \quad \text{in } K_0(\mathfrak{C}).$$

This corresponds to

$$\sum_{0 \leq k < 2^n} q'(n,k)p_{(n,k)}(X) = \varrho_n(X). \tag{4.4}$$

One can give a representation of $K_0(\mathfrak{C})$ where the injective maps β_n in (4.2) are replaced by inclusions $\iota_n(p) = p$. Define

$$\phi_{(n,k)}(X) = \frac{p_{(n,k)}(X^{1/2^n})}{\varrho_{(n,k)}(X^{1/2^n})} = \begin{cases} \dfrac{1}{\prod_{j=1}^n (X^{-1/2^j} + 1 + X^{1/2^j})} & \text{if } k = 0, \\ \dfrac{X^{k/2^n} + X^{-k/2^n}}{\prod_{j=1}^n (X^{-1/2^j} + 1 + X^{1/2^j})} & \text{if } 1 \leq k < 2^n, \end{cases}$$

and consider the additive abelian group

$$\mathcal{R}_n := \Big\{ \sum_{0 \leq k < 2^n} c_k \phi_{(n,k)} : c_k \in \mathbb{Z} \Big\}.$$

Equalities (4.1) become

$$\begin{cases} \phi_{(n+1,0)} + \phi_{(n+1,1)} = \phi_{(n,0)}, \\ \phi_{(n+1,2k-1)} + \phi_{(n+1,2k)} + \phi_{(n+1,2k+1)} = \phi_{(n,k)}, \quad 1 \leq k < 2^n, \end{cases}$$

and show that $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$ and that the diagram

$$\begin{tikzcd} K_0(\mathfrak{C}_n) \arrow[r, "\psi_n"] \arrow[d, "\alpha_n"'] & \mathcal{R}_n \arrow[d, hook, "\iota_n"] \\ K_0(\mathfrak{C}_{n+1}) \arrow[r, "\psi_{n+1}"'] & \mathcal{R}_{n+1} \end{tikzcd}$$

is commuting, where $\psi([e_{(n,k)}]) = \phi_{(n,k)}$. Therefore $K_0(\mathfrak{C}) = \mathcal{R} := \cup_n \mathcal{R}_n$. Taking $X = \mathrm{e}^Y$, we see that $K_0(\mathfrak{C})$ can be viewed as the $\mathbb{Z}$-linear span of $\widetilde{\phi}_{(n,k)}$, $n \geq 0$,

$0 \le k < 2^n$, where

$$\widetilde{\phi}_{(n,k)}(Y) = \begin{cases} \dfrac{1}{\prod_{j=1}^{n}(1+2\cosh(Y/2^j))} & \text{if } k=0, \\ \dfrac{2\cosh(kY/2^n)}{\prod_{j=1}^{n}(1+2\cosh(Y/2^j))} & \text{if } 1 \le k < 2^n. \end{cases}$$

One can certainly replace Y by iY and use cos instead of cosh.

5. Traces on $\mathfrak{A}$

We augment the diagram $\mathcal{G} = D(\mathfrak{A})$ into $\widetilde{\mathcal{G}}$, by adding a $(-1)^{\mathrm{st}}$ floor with only one vertex $\star = (-1,0)$ connected to both $(0,0)$ and $(0,1)$. Traces τ on $\mathfrak{A}$ are in one-to-one correspondence (cf., e.g., [13, Section 3.6]) with families $\alpha^\tau = (\alpha^\tau_{(n,k)})$ of numbers in $[0,1]$, $n \ge -1$, $0 \le k \le 2^n$, such that

$$\begin{cases} \alpha^\tau_\star = 1, & \\ \alpha^\tau_{(n,0)} = \alpha^\tau_{(n+1,0)} + \alpha^\tau_{(n+1,1)} & \text{if } n \ge -1, \\ \alpha^\tau_{(n,2^n)} = \alpha^\tau_{(n+1,2^{n+1})} + \alpha^\tau_{(n+1,2^{n+1}-1)} & \text{if } n \ge 0, \\ \alpha^\tau_{(n,k)} = \alpha^\tau_{(n+1,2k-1)} + \alpha^\tau_{(n+1,2k)} + \alpha^\tau_{(n+1,2k+1)} & \text{if } n \ge 1,\ 0 < k < 2^n. \end{cases}$$

An inspection of $\widetilde{\mathcal{G}}$ shows that such a family α^τ is uniquely determined by the numbers $\alpha^\tau_{(n,k)}$ with odd k. Let $\mathcal{T}$ denote the diagram obtained by removing the memory in $\widetilde{\mathcal{G}}$. Its set of vertices $V(\mathcal{T})$ consists of $\star$ and (n,k) with $n \ge 0$ and odd k. For $v = (n,k)$ define $Lv = (n+1, 2k-1)$ if $n \ge 0$, $0 < k \le 2^n$, and $Rv = (n+1, 2k+1)$ if $n \ge -1$, $0 \le k < 2^n$.

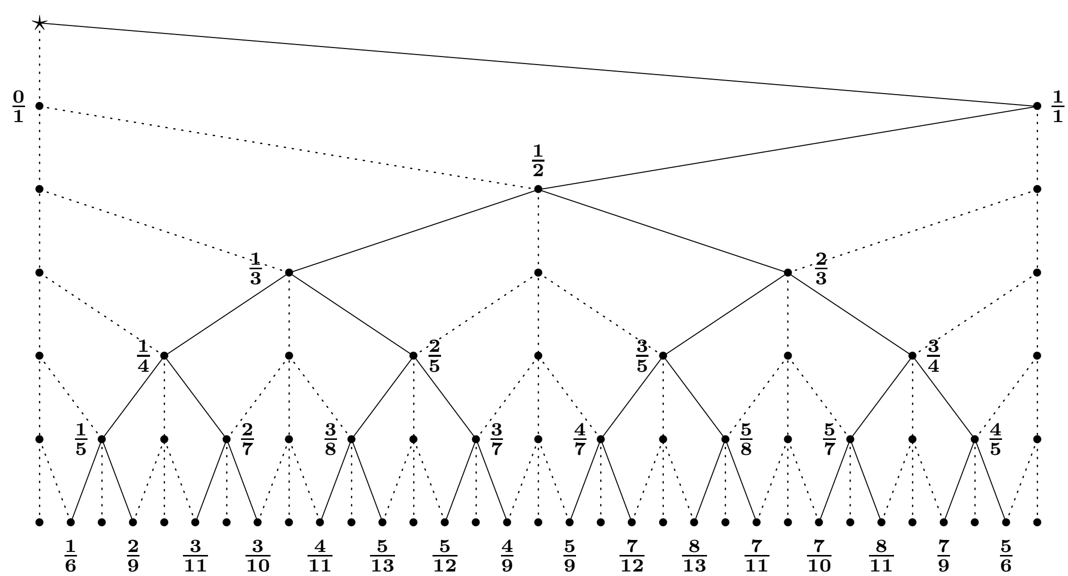


Figure 13. The diagram $\mathcal{T}$

Given α^τ_v, $v=(n,k)\in V(\mathcal{T})$, define recursively for $r\geq 1$

$$\begin{cases} \alpha^\tau_{(n+r,0)} = \alpha^\tau_{(n+r-1,0)} - \alpha^\tau_{(n+r,1)} & \text{if } n\geq -1,\\ \alpha^\tau_{(n+r,2^{n+r})} = \alpha^\tau_{(n+r-1,2^{n+r-1})} - \alpha^\tau_{(n+r,2^{n+r}-1)} & \text{if } n\geq 0,\\ \alpha^\tau_{(n+r,2^r k)} = \alpha^\tau_{(n+r-1,2^{r-1}k)} - \alpha^\tau_{(n+r,2^r k-1)} - \alpha^\tau_{(n+r,2^r k+1)} & \text{if } n\geq 1, \end{cases}$$

or equivalently

$$\begin{cases} \alpha^\tau_{(n,0)} = \alpha^\tau_\star - \sum\limits_{j=0}^{n} \alpha^\tau_{(j,1)} = \alpha^\tau_\star - \sum\limits_{j=0}^{n} \alpha^\tau_{L^j R\star} & \text{if } n\geq 0,\\ \alpha^\tau_{(n,2^n)} = \alpha^\tau_{(0,1)} - \sum\limits_{j=1}^{n} \alpha^\tau_{(j,2^j-1)} = \alpha^\tau_{(0,1)} - \sum\limits_{j=1}^{n} \alpha^\tau_{R^{j-1}L(0,1)} & \text{if } n\geq 1,\\ \alpha^\tau_{(n+r,2^r k)} = \alpha^\tau_{(n,k)} - \sum\limits_{j=1}^{r} \left(\alpha^\tau_{(n+j,2^j k-1)} + \alpha^\tau_{(n+j,2^j k+1)}\right) & \\ \qquad\qquad = \alpha^\tau_{(n,k)} - \sum\limits_{j=1}^{r} \left(\alpha^\tau_{R^{j-1}L(n,k)} + \alpha^\tau_{L^{j-1}R(n,k)}\right) & \text{if } n\geq 2. \end{cases} \tag{5.1}$$

There is an obvious order relation on $V(\mathcal{T})$ defined by $(n,k_n)\preceq(n',k'_n)$ if $n\leq n'$ and there is a chain of vertices $(n,k_n),\dots,(n',k'_n)$ such that $(n+i,k_{n+i})$ is connected to $(n+i+1,k_{n+i+1})$, i.e. $k_{n+i+1}-2k_{n+i}=\pm 1$. A function $f:V(\mathcal{T})\to\mathbb{R}$ is monotonically decreasing if $f(v_1)\geq f(v_2)$ whenever $v_1\preceq v_2$ in $V(\mathcal{T})$. For each vertex $v=(n,k)\in V(\mathcal{T})$, let

$$\mathcal{C}_v = \begin{cases} \{L^j R\star : j\geq 0\} & \text{if } v=\star,\\ \{R^{j-1}L(0,1) : j\geq 1\} & \text{if } v=(0,1),\\ \{R^{j-1}Lv : j\geq 1\}\cup\{L^{j-1}Rv : j\geq 1\} & \text{if } v\in V(\mathcal{T})\setminus\{\star,(0,1)\}, \end{cases} \tag{5.2}$$

denote the set of vertices in $V(\mathcal{T})$ neighboring the vertical infinite segment originating at v. As a result of (5.1) and of non-negativity of α^τ we have

Proposition 13. *There is a one-to-one correspondence between traces on $\mathfrak{A}$ and functions $\phi:V(\mathcal{T})\to[0,1]$ such that $\phi(\star)=1$ and*

$$\phi(v)\geq \sum_{w\in\mathcal{C}_v}\phi(w),\quad \forall v\in V(\mathcal{T}). \tag{5.3}$$

Note that a function satisfying (5.3) is necessarily monotonically decreasing.

One can give a description of the set $\mathcal{C}_v$ using the one-to-one correspondence $v\mapsto r(v)$ between the sets $V(\mathcal{T})$ and $\mathbb{Q}\cap[0,1]$ (see Figure 14). Any number in $\mathbb{Q}\cap(0,1)$ can be uniquely represented as a (reduced) continued fraction $[a_1,\dots,a_t]$ with $a_t\geq 2$. It is not hard to notice and prove that, for any $v\in V(\mathcal{T})$ with $r(v)=[a_1,\dots,a_t]$, $a_t\geq 2$, we have

$$\begin{aligned} r(Lv) &= \begin{cases} [a_1,\dots,a_{t-1},a_t-1,2] & \text{if } t \text{ even},\\ [a_1,\dots,a_{t-1},a_t+1] & \text{if } t \text{ odd}, \end{cases}\\ r(Rv) &= \begin{cases} [a_1,\dots,a_{t-1},a_t+1] & \text{if } t \text{ even},\\ [a_1,\dots,a_{t-1},a_t-1,2] & \text{if } t \text{ odd}. \end{cases} \end{aligned} \tag{5.4}$$

As a result of (5.2) and (5.4) we have

$$\{r(w): w\in\mathcal{C}_v\} = \{[a_1,\dots,a_{t-1},a_t-1,1,k]: k\geq 1\}\cup\{[a_1,\dots,a_{t-1},a_t,k]: k\geq 1\},$$

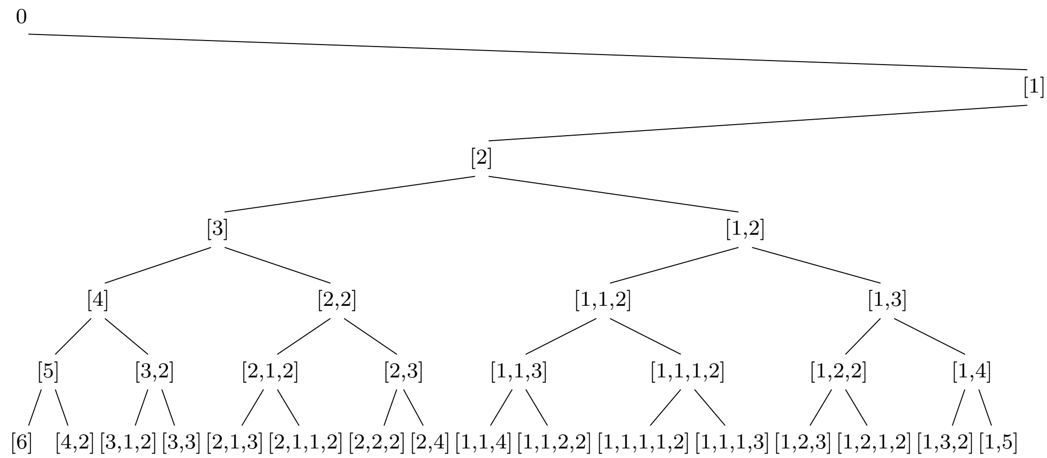


FIGURE 14. The diagram $\mathcal{T}$ in the continued fraction representation

which shows in conjunction with Proposition 13 that there is a one-to-one correspondence between traces on $\mathfrak{A}$ and maps $\phi : \mathbb{Q} \cap [0,1] \to [0,1]$ which satisfy

$$\begin{cases} 1 = \phi(0) \geq \sum\limits_{k=1}^{\infty} \phi\big(\frac{1}{k}\big), \quad \phi(1) \geq \sum\limits_{k=1}^{\infty} \phi\big(\frac{k}{k+1}\big), \\ \phi([a_1,\dots,a_t]) \geq \sum\limits_{k=1}^{\infty} \big(\phi([a_1,\dots,a_{t-1},a_t-1,1,k]) + \phi([a_1,\dots,a_{t-1},a_t,k]\big), \quad a_t \geq 2. \end{cases}$$

6. Generators, relations, and braiding

We shall use the path algebra model for AF algebras as in [15, Section 2.3.11] and [13, Section 2.9]. Here however a monotone increasing path ξ will be encoded by the sequence (ξ_n) where ξ_n gives the "horizontal coordinate" of the vertex at floor n, instead of its edges. To use this model we again augment the diagram $\mathcal{G} = D(\mathfrak{A})$ into $\widetilde{\mathcal{G}}$.

Denote by Ω the (uncountable) set of monotone increasing paths starting at $\star$. Let $\Omega_{[r}$ denote the set of infinite monotone increasing paths starting on the r^{th} floor of $\widetilde{\mathcal{G}}$, $\Omega_{r]}$ the set of monotone increasing paths that connect $\star$ with a vertex on the r^{th} floor, and $\Omega_{[r,s]}$ the set of monotone increasing paths starting on the r^{th} floor and ending on the s^{th} floor. Let $\xi_r] \in \Omega_{r]}$, $\xi_{[r,s]} \in \Omega_{[r,s]}$, $\xi_{[s} \in \Omega_{[s}$ denote the natural truncations of a path $\xi \in \Omega$. By $\xi \circ \eta$ we denote the natural concatenation of two paths $\xi \in \Omega_{r]}$ and $\eta \in \Omega_{[r}$ with $\xi_r = \eta_r$. Consider the set R_r of pairs of paths $(\xi,\eta) \in \Omega_{r]} \times \Omega_{r]}$ with the same endpoint $\xi_r = \eta_r$. For each $(\xi,\eta) \in R_r$ the mapping

$$\Omega \ni \omega \mapsto T_{\xi,\eta}\omega = \delta(\eta,\omega_{r]})\xi \circ \omega_{[r} \in \Omega,$$

extends to a linear operator on the $\mathbb{C}$-linear space $\mathbb{C}\Omega$ with basis Ω, and also to a bounded operator $T_{\xi,\eta} : \ell^2(\Omega) \to \ell^2(\Omega)$ with $\|T_{\xi,\eta}\| = 1$. We have $\mathfrak{A} = \overline{\cup_{r\geq 1}\mathfrak{A}_r}$ where the linear span $\mathfrak{A}_r$ of the operators $T_{\xi,\eta}$, $(\xi,\eta) \in R_r$, forms a finite dimensional C^*-algebra as a result of

$$T^*_{\eta,\xi} = T_{\xi,\eta}, \qquad T_{\xi,\eta}T_{\xi',\eta'} = \delta(\eta,\xi')T_{\xi,\eta'}, \qquad \sum_{\xi\in\Omega_{r]}} T_{\xi,\xi} = 1,$$

and the inclusion $\mathfrak{A}_r \overset{\iota_r}{\hookrightarrow} \mathfrak{A}_{r+1}$ is given by

$$\iota_r(T_{\xi,\eta}) = \sum_{\substack{\lambda\in\Omega_{[r,r+1]} \\ \lambda_r=\xi_r(=\eta_r)}} T_{\xi\circ\lambda,\eta\circ\lambda}.$$

This model is employed to give a presentation by generators and relations of the C^*-algebra $\mathfrak{A}$ in the spirit of the presentation of the GICAR algebra from [13, Example 2.23]. We also construct two families of projections that satisfy commutation relations reminiscent of the Temperley-Lieb relations. Consider the following elements in $\mathfrak{A}$:

(1) the projection e_n in $\mathfrak{A}_{n-1,n} \subseteq \mathfrak{A}_n$ onto the linear space of edges from N (north) to SW (south-west), $n \geq 1$.
(2) the projection f_n in $\mathfrak{A}_{n-1,n} \subseteq \mathfrak{A}_n$ onto the linear span of edges from N to SE, $n \geq 0$.
(3) the projection $g_n = 1 - e_n - f_n$ in $\mathfrak{A}_{n-1,n} \subseteq \mathfrak{A}_n$ onto the linear span of edges from N to S, $n \geq 0$.
(4) the partial isometry $v_n \in \mathfrak{A}_{n-1,n+1} \subseteq \mathfrak{A}_{n+1}$ with initial support $v_n^* v_n = \widetilde{e}_n = g_n f_{n+1}$ and final support $v_n v_n^* = \widetilde{f}_n = f_n e_{n+1}$, which flips paths in the diamonds of shape N-S-SE-NE, $n \geq 0$.
(5) the partial isometry $w_n \in \mathfrak{A}_{n-1,n+2} \subseteq \mathfrak{A}_{n+1}$ with initial support $w_n^* w_n = \widetilde{e'}_n = g_n e_{n+1}$ and final support $w_n w_n^* = \widetilde{f}'_n = e_n f_{n+1}$, which flips paths in the diamonds of shape N-S-SW-NW, $n \geq 1$.

The AF-algebra $\mathfrak{A}$ is generated by the set $\mathfrak{G} = \{e_n\}_{n\geq 1} \cup \{f_n\}_{n\geq 0} \cup \{v_n\}_{n\geq 0} \cup \{w_n\}_{n\geq 1}$.

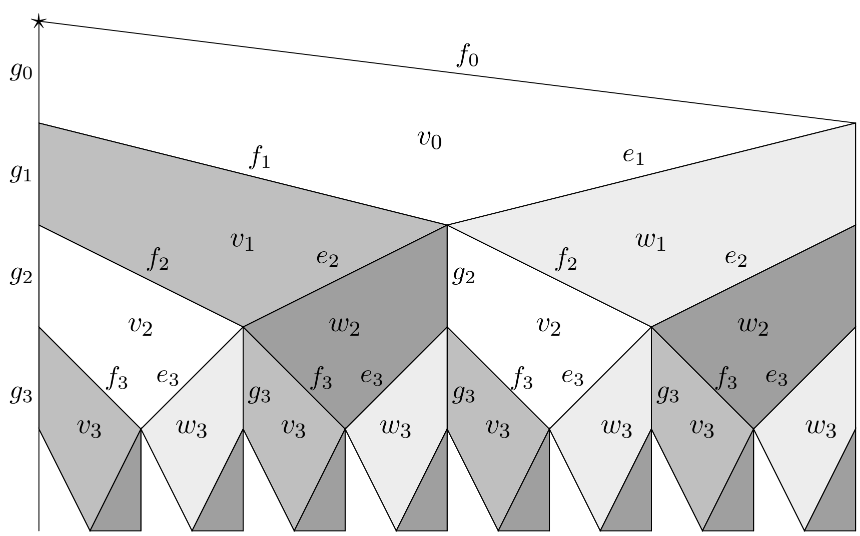


FIGURE 15. The generators of $\mathfrak{A}$

Straightforward commutation relations arise since elements defined by edges that reach up to floor $\leq r$ commute with elements defined by edges between the r^{th} and the s^{th} floors with $r < s$, as a result of $[\mathfrak{A}_r, \mathfrak{A}'_r \cap \mathfrak{A}_s] = 0$. For instance v_s commutes with e_r, f_r, g_r if $r \leq s-1$ or $r \geq s+2$, and $[v_s, v_r] = [v_s, v_r^*] = [v_s, w_r] = [v_s, w_r^*] = 0$ if $|r-s| \geq 2$. Besides, the elements of $\mathfrak{G}$ satisfy the following commutation relations:

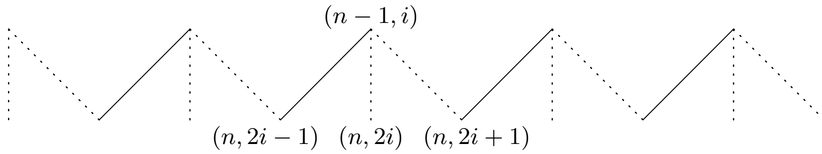


FIGURE 16. Support of projection e_n

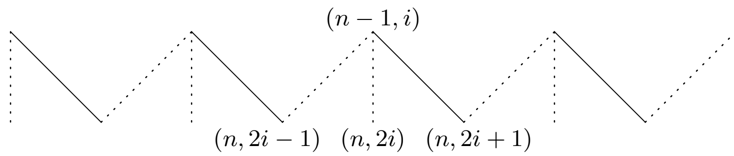


FIGURE 17. Support of projection f_n

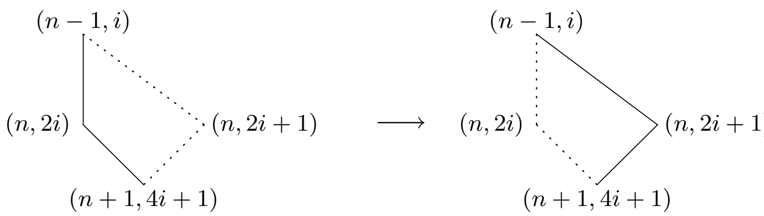


FIGURE 18. The partial isometry $v_n : g_n f_{n+1} \mapsto f_n e_{n+1}$

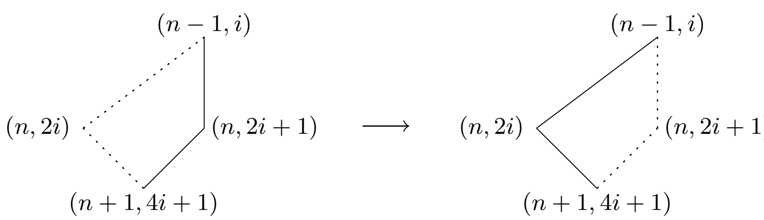


FIGURE 19. The partial isometry $w_n : g_n e_{n+1} \mapsto e_n f_{n+1}$

(R1) $e_n^2 = e_n^* = e_n$, $f_n^2 = f_n^* = f_n$, $g_n^2 = g_n^* = g_n$, $e_n + f_n + g_n = 1$;
e_n, f_m, g_k mutually commute.
(R2) $(1-f_n)v_n = (1-e_{n+1})v_n = 0$, $v_n(1-g_n) = v_n(1-f_{n+1}) = 0$.
$(1-e_n)w_n = (1-f_{n+1})w_n = 0$, $w_n(1-g_n) = w_n(1-e_{n+1}) = 0$.
(R3) $v_n g_n = f_n v_n$, $v_n f_{n+1} = e_{n+1} v_n$, $w_n g_n = e_n w_n$, $w_n e_{n+1} = f_{n+1} w_n$.
(R4) $v_n^* v_n = g_n f_{n+1}$, $v_n v_n^* = f_n e_{n+1}$, $w_n^* w_n = g_n e_{n+1}$, $w_n w_n^* = e_n f_{n+1}$.

As a result of (R1)–(R4) we also get

$$\begin{aligned} &v_{n+1}v_n = v_n^2 = v_{n\pm 1}v_n^* = v_{n\pm 1}^* v_n = 0, \\ &w_{n+1}w_n = w_n^2 = w_{n\pm 1}w_n^* = w_{n\pm 1}^* w_n = 0, \\ &v_n w_n = v_{n\pm 1}w_n = w_n v_n = w_{n\pm 1}v_n = 0, \\ &v_n w_n^* = v_{n\pm 1}w_n^* = v_n^* w_n = v_n^* w_{n-1} = 0. \end{aligned} \tag{6.1}$$

The only non-zero products ab with $a \in \{v_n, v_n^*, w_n, w_n^*\}$ and $b \in \{v_{n+1}, v_{n+1}^*, w_{n+1}, w_{n+1}^*\}$ are $v_n v_{n+1}$, $w_n w_{n+1}$, $w_n^* v_{n+1}$, and $v_n^* w_{n+1}$.

Let B_n denote Artin's braid group generated by $\sigma_1, \dots, \sigma_{n-1}$ with relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. Relations (6.1) show in particular that the partial isometries v_{i-1}, respectively w_i, satisfy these braid relations.

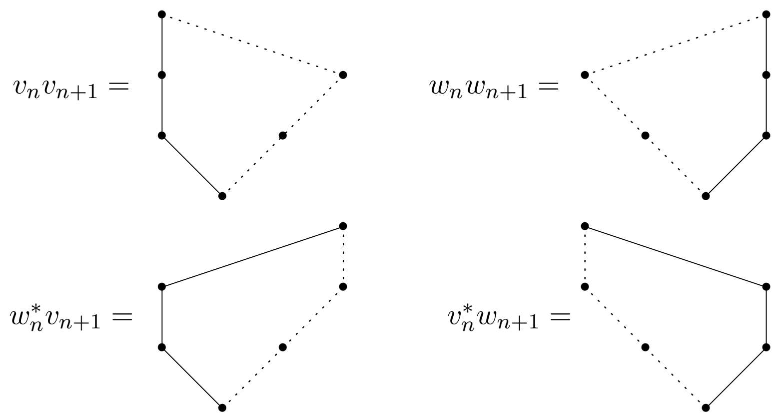


FIGURE 20. The partial isometries $v_n v_{n+1} : g_n g_{n+1} f_{n+2} \mapsto f_n e_{n+1} e_{n+2}$, $w_n w_{n+1} : g_n g_{n+1} e_{n+2} \mapsto e_n f_{n+1} f_{n+2}$, $w_n^* v_{n+1} : e_n g_{n+1} f_{n+2} \mapsto g_n e_{n+1} e_{n+2}$, $v_n^* w_{n+1} : f_n g_{n+1} e_{n+2} \mapsto g_n f_{n+1} f_{n+2}$

Taking $R_n(\lambda) := 1 + \lambda v_n$, the equalities

$$v_n^2 = 0, \qquad v_n v_{n\pm 1} v_n = 0, \tag{6.2}$$

yield the Yang-Baxter type relation

$$R_n(\lambda) R_{n+1}(\lambda+\mu) R_n(\mu) = R_{n+1}(\mu) R_n(\lambda+\mu) R_{n+1}(\lambda). \tag{6.3}$$

By analogy with the construction of Temperley-Lieb-Jones projections in the GICAR algebra (cf., e.g., [13] or [15]) for each $\lambda > 0$ we put $\tau = \frac{\lambda}{(1+\lambda)^2} \in (0, \frac{1}{4}]$ and consider

$$E_n = \frac{1}{1+\lambda} \left(v_n^* v_n + \sqrt{\lambda}\, v_n + \sqrt{\lambda}\, v_n^* + \lambda v_n v_n^* \right) \in \mathfrak{A}, \quad n \geq 0, \tag{6.4}$$

$$F_n = \frac{1}{1+\lambda} \left(w_n^* w_n + \sqrt{\lambda}\; w_n + \sqrt{\lambda}\; w_n^* + \lambda w_n w_n^* \right) \in \mathfrak{A}, \quad n \geq 1. \tag{6.5}$$

Proposition 14. *The elements E_n and F_n define (self-adjoint) projections in the AF algebra $\mathfrak{A}$ satisfying the braiding relations*

$$E_n F_n = F_n E_n = 0, \tag{6.6}$$

$$[E_n, E_m] = [F_n, F_m] = [E_n, F_m] = 0 \quad \text{if } |n-m| \geq 2, \tag{6.7}$$

$$E_n E_{n+1} E_n = \tau E_n e_{n+2}, \quad E_{n+1} E_n E_{n+1} = \tau E_{n+1} g_n, \tag{6.8}$$

$$F_n F_{n+1} F_n = \tau F_n f_{n+2}, \quad F_{n+1} F_n F_{n+1} = \tau F_{n+1} g_n, \tag{6.9}$$

$$E_n F_{n+1} E_n = \lambda\tau E_n f_{n+2}, \quad F_n E_{n+1} F_n = \lambda\tau F_n e_{n+2}, \tag{6.10}$$

$$E_{n+1} F_n E_{n+1} = \lambda\tau E_{n+1} e_n, \quad F_{n+1} E_n F_{n+1} = \lambda\tau F_{n+1} f_n, \tag{6.11}$$

$$E_n E_{n+1} F_n = E_n F_{n+1} F_n = E_{n+1} E_n F_{n+1} = E_{n+1} F_n F_{n+1} = 0, \tag{6.12}$$

$$F_n E_{n+1} E_n = F_n F_{n+1} E_n = F_{n+1} E_n E_{n+1} = F_{n+1} F_n E_{n+1} = 0. \tag{6.13}$$

Proof. The initial and final projections of the partial isometry v_n are orthogonal, thus E_n defines a projection in $\mathfrak{A}_n$ for every $\lambda \geq 0$. A similar property holds for F_n, which is seen to be orthogonal to E_n. The commutation relations (6.7) are obvious because

v_{n+2} and w_{n+2} commute with all elements in $\mathfrak{A}_{n+1}$, including E_n and F_n. By (6.1) we have $v_n^* E_{n+1} = v_n v_{n+1}^* = 0$, leading to

$$E_n E_{n+1} = \frac{\sqrt{\lambda}}{(1+\lambda)^2} \, \big(v_n^* v_n + \sqrt{\lambda}\; v_n \big) \big(v_{n+1} + \sqrt{\lambda}\; v_{n+1} v_{n+1}^* \big), \tag{6.14}$$

and also

$$E_{n+1} E_n = (E_n E_{n+1})^* = \frac{\sqrt{\lambda}}{(1+\lambda)^2} \, \big(v_{n+1}^* + \sqrt{\lambda}\; v_{n+1} v_{n+1}^* \big) \big(v_n^* v_n + \sqrt{\lambda}\; v_n^* \big). \tag{6.15}$$

From (6.14) and $v_{n+1} E_n = v_{n+1}^* v_n = 0$ we have

$$\begin{aligned} E_n E_{n+1} E_n &= \frac{\lambda}{(1+\lambda)^3} \, \big(v_n^* v_n + \sqrt{\lambda}\; v_n \big) v_{n+1} v_{n+1}^* \big(v_n^* v_n + \sqrt{\lambda}\; v_n^* \big) \\ &= \frac{\lambda}{(1+\lambda)^3} \, \big(\widetilde{e}_n + \sqrt{\lambda}\; v_n \big) \widetilde{f}_{n+1} \big(\widetilde{e}_n + \sqrt{\lambda}\; v_n^* \big). \end{aligned} \tag{6.16}$$

But $\widetilde{e}_n \widetilde{f}_{n+1} \widetilde{e}_n = \widetilde{e}_n \widetilde{f}_{n+1} = g_n f_{n+1} e_{n+1} = \widetilde{e}_n e_{n+2}$, $v_n \widetilde{f}_{n+1} \widetilde{e}_n = v_n \widetilde{e}_n e_{n+1} e_{n+2} = v_n e_{n+2}$ (and because $[e_{n+2}, v_n] = 0$ this also gives $\widetilde{e}_n \widetilde{f}_{n+2} v_n^* = v_n^* e_{n+2}$), and $v_n \widetilde{f}_{n+1} v_n^* = v_n f_{n+1} e_{n+2} v_n^* = v_n f_{n+1} v_n^* e_{n+2} = v_n g_n f_{n+1} v_n^* e_{n+2} = v_n v_n^* e_{n+2}$, which we insert in (6.16) to get

$$E_n E_{n+1} E_n = \tau E_n e_{n+2}.$$

From (6.15) and $v_n^* E_{n+1} = v_n^* v_{n+1}^* = 0$ we find

$$E_{n+1} E_n E_{n+1} = \frac{\lambda}{(1+\lambda)^3} \, \big(v_{n+1}^* + \sqrt{\lambda}\; \widetilde{f}_{n+1} \big) \widetilde{e}_n \big(v_{n+1} + \sqrt{\lambda}\; \widetilde{f}_{n+1} \big). \tag{6.17}$$

As a result of $[g_n, v_{n+1}] = 0$ and $(1 - f_{n+1}) v_{n+1} = 0$ we have $v_{n+1}^* \widetilde{e}_n v_{n+1} = \widetilde{e}_{n+1} g_n$. It is also plain that $\widetilde{f}_{n+1} \widetilde{e}_n \widetilde{f}_{n+1} = \widetilde{f}_{n+1} \widetilde{e}_n = \widetilde{f}_{n+1} g_n$, $\widetilde{f}_{n+1} \widetilde{e}_n v_{n+1} = \widetilde{f}_{n+1} g_n v_{n+1} = \widetilde{f}_{n+1} v_{n+1} g_n = v_{n+1} g_n$, and $v_{n+1}^* \widetilde{e}_n \widetilde{f}_{n+1} = v_{n+1}^* \widetilde{f}_{n+1} g_n = v_{n+1}^* g_n$. Together with (6.17) these equalities yield

$$E_{n+1} E_n E_{n+1} = \tau E_{n+1} g_n.$$

Equalities (6.9)–(6.12) are checked in a similar way. (6.13) follows by taking adjoints in (6.12). □

Acknowledgments

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